

STRONG RENEWAL THEOREMS WITH INFINITE MEAN

BY

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Abstract. Let F be a nonarithmetic probability distribution on $(0, \infty)$ and suppose $1 - F(t)$ is regularly varying at ∞ with exponent α , $0 < \alpha \leq 1$. Let $U(t) = \sum F^{n*}(t)$ be the renewal function. In this paper we first derive various asymptotic expressions for the quantity $U(t+h) - U(t)$ as $t \rightarrow \infty$, $h > 0$ fixed. Next we derive asymptotic relations for the convolution $U * z(t)$, $t \rightarrow \infty$, for a large class of integrable functions z . All of these asymptotic relations are expressed in terms of the truncated mean function $m(t) = \int_0^t [1 - F(x)] dx$, t large, and appear as the natural extension of the classical strong renewal theorem for distributions with finite mean. Finally in the last sections of the paper we apply the special case $\alpha = 1$ to derive some limit theorems for the distributions of certain waiting times associated with a renewal process.

1. Principal theorems. Let F be a probability measure concentrated on $[0, \infty)$ ⁽²⁾ and let U be the associated renewal measure defined for any measurable set I by

$$(1.1) \quad U\{I\} = \sum_{n=0}^{\infty} F^{n*}\{I\}$$

where F^{n*} denotes the n -fold convolution of F with itself (F^{0*} is the probability measure concentrated at the origin). The series (1.1) converges to a finite number for every bounded I . (For this and other elementary properties of U see [3, VI. 6]; for a probabilistic interpretation of U see §9 in this paper.) We write $U(x)$ for $U\{[0, x]\}$ and we shall henceforth ignore the distinction between U the measure and U the function. (This convention applies to other measures as well.)

The main results of this paper deal primarily with the differences $U(t+h) - U(t)$ for $h > 0$ fixed, and $t \rightarrow \infty$. The principal assumption is that F has the form

$$(1.2) \quad 1 - F(t) = t^{-\alpha} L(t), \quad t > 0,$$

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⁽²⁾ We assume, however, that not all the mass is at the origin.

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where $0 \leq \alpha \leq 1$ (fixed) and L is a slowly varying function⁽³⁾. Unless otherwise indicated, we also assume F is nonarithmetic; that is, we exclude the possibility that F concentrates the entire mass on the multiples of some positive real number. For $\alpha \neq 1$, the arithmetic versions of Theorems 1 and 2 below were treated by A. Garsia and J. Lamperti, [5] (nothing was known in the case $\alpha = 1$). See §2(ii) for further discussion. Define the "truncated mean" function

$$(1.3) \quad m(t) = \int_0^t (1 - F(x)) dx = t(1 - F(t)) + \int_0^t xF\{dx\}.$$

THEOREM 1. Let F satisfy (1.2) with $\frac{1}{2} < \alpha \leq 1$. Then for every $h > 0$ and as $t \rightarrow \infty$

$$(1.4) \quad U(t+h) - U(t) \sim C_\alpha h / m(t)$$

where $C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$.

THEOREM 2. If $0 < \alpha \leq \frac{1}{2}$ then

$$(1.5) \quad \liminf_{t \rightarrow \infty} m(t)(U(t+h) - U(t)) = C_\alpha h.$$

REMARK. When $\alpha \neq 1$, $m(t) \sim (1-\alpha)^{-1} t^{1-\alpha} L(t)$, $t \rightarrow \infty$ (see Lemma 1, §3) and $\Gamma(\alpha)\Gamma(2-\alpha) = \pi(1-\alpha) \csc \pi\alpha$. It follows that (1.4) is equivalent to

$$(1.6) \quad \lim_{t \rightarrow \infty} t^{1-\alpha} L(t)(U(t+h) - U(t)) = \frac{\sin \pi\alpha}{\pi} h.$$

The results of Theorems 2, 3, and 4 may be restated in an analogous fashion.

Let z be a nonnegative function on $[0, \infty)$. For $h > 0$ write

$$\sigma^+ = h \sum_{k=1}^{\infty} \sup \{z(x) : (k-1)h \leq x < kh\}$$

and similarly define σ_- with inf in place of sup. Following Feller [3, p. 348], we say that z is *directly Riemann integrable* (dri) if the series defining the upper sum σ^+ converges and $\sigma^+ - \sigma_- \rightarrow 0$ as $h \rightarrow 0$. It follows immediately that a dri function is bounded, measurable and (Lebesgue) integrable.

THEOREM 3. Let z be a nonnegative dri function on $[0, \infty)$ which satisfies

$$(1.7) \quad z(t) = O(1/t), \quad t > 0.$$

If F has the form (1.2) with $\frac{1}{2} < \alpha \leq 1$ then

$$(1.8) \quad \int_0^t z(t-y)U\{dy\} \sim \frac{C_\alpha}{m(t)} \int_0^\infty z(x) dx.$$

⁽³⁾ A measurable ultimately positive function L on $[0, \infty)$ is regularly varying with exponent ρ if as $t \rightarrow \infty$, $L(xt)/L(t) \rightarrow x^\rho$ for all $x > 0$. When $\rho = 0$, i.e., $L(xt)/L(t) \rightarrow 1$, we also say L is slowly varying. We assume as known the various properties of slowly varying functions as described in [3, pp. 272-274], or in [6]. Note that the function L in (1.2) must be bounded on bounded subintervals of $[0, \infty)$.

THEOREM 4. *Let $z \geq 0$ be a dri function (not necessarily satisfying (1.7)). If F satisfies (1.2) with $\alpha \neq 0$ then*

$$(1.9) \quad \liminf_{t \rightarrow \infty} m(t) \int_0^t z(t-y)U\{dy\} = C_\alpha \int_0^\infty z(x) dx.$$

REMARKS. 1. Define a complex valued z to be dri if $|z|$ is dri as defined above. With this definition it follows readily from Theorem 3 that (1.8) holds for any dri z satisfying (1.7).

2. Any piecewise continuous function on $[0, \infty)$ vanishing off a compact interval is dri and certainly satisfies (1.7). In particular, taking $z(x)=1$ for $0 \leq x \leq h$, and $z(x)=0$ elsewhere we have by (1.8)

$$U(t+h) - U(t) = \int_0^{t+h} z(t+h-x)U\{dx\} \sim \frac{C_\alpha h}{m(t+h)} \sim C_\alpha \frac{h}{m(t)}$$

as $t \rightarrow \infty$. (That $m(t+h) \sim m(t)$, $t \rightarrow \infty$, h fixed, follows easily from monotonicity and regular variation of m , see Lemma 1.) Thus Theorem 3 is equivalent to Theorem 1 (we use Theorem 1 to prove Theorem 3). Similarly Theorem 4 (with $0 < \alpha \leq \frac{1}{2}$) is equivalent to Theorem 2.

For a generalization of (1.8) to nonintegrable but regularly varying z see §2(iii).

§§3–8 of this paper are concerned with the proofs of Theorems 1–4. In §9 we give an application of the special case $\alpha=1$ to obtain some curious limit theorems for the spent and residual waiting times of a renewal process.

2. Notes. (i) Let m and U be defined as in §1 and let $\hat{m}$ and $\hat{U}$ be their Laplace transforms:

$$\hat{m}(\lambda) = \int_0^\infty e^{-\lambda x}(1-F(x)) dx, \quad \hat{U}(\lambda) = \int_0^\infty e^{-\lambda x}U\{dx\}.$$

If in addition $\hat{F}$ is the transform of F then by (1.1) and (1.3)

$$\hat{m}(\lambda) = \frac{1-\hat{F}(\lambda)}{\lambda}, \quad \hat{U}(\lambda) = \frac{1}{1-\hat{F}(\lambda)}$$

and hence $\hat{U}(\lambda)\hat{m}(\lambda) = 1/\lambda$. Using this relation and Karamata's Tauberian theorem, [3, p. 420], we conclude the following:

THEOREM 5. *Let $0 \leq \alpha \leq 1$. Each of statements (a) and (b) which follow implies the other and both imply the asymptotic relation (2.1).*

- (a) *m is regularly varying with exponent $1-\alpha$.*
- (b) *U is regularly varying with exponent α .*

$$(2.1) \quad U(t) \sim [\Gamma(\alpha+1)\Gamma(2-\alpha)]^{-1}(t/m(t)).$$

By Lemma 1 statement (a) is true when F satisfies (1.2). (The converse is also true provided $\alpha \neq 1$; if (a) is true for some $0 \leq \alpha < 1$, then (1.2) holds for some slowly

varying L , cf. [3, p. 422].) When $\alpha \neq 1$ in (1.2) we see as in the remark following Theorem 2 that (2.1) is equivalent to

$$(2.2) \quad U(t) \sim \frac{\sin \pi \alpha}{\pi \alpha} \frac{t^\alpha}{L(t)}, \quad t \rightarrow \infty,$$

(when $\alpha = 0$, $(\sin \pi \alpha)/\pi \alpha \equiv 1$). For a proof of (2.2) when $0 < \alpha < 1$ cf. [3, p. 446]. See also Teugels [10]. When $\frac{1}{2} < \alpha \leq 1$ (2.1) may also be derived from Theorem 1 (1.4). We shall not do this however. Theorem 1 cannot be proved from (2.1).

(ii) Let F be an arithmetic distribution on $(0, \infty)$ which we suppose, without loss of generality, has span 1. (A distribution has span $b > 0$ if it is concentrated on the multiples of b and b is the largest such number.) The renewal measure U defined by (1.1) is also arithmetic with span 1. Denote by f_n and u_n the mass assigned to the integer n by F and U . If F satisfies (1.2), i.e.,

$$1 - F(n) = \sum_{k=n+1}^{\infty} f_k = n^{-\alpha} L(n)$$

for some $0 < \alpha < 1$ and slowly varying L , then (Lamperti-Garsia, 1962) for $\frac{1}{2} < \alpha < 1$

$$(2.3) \quad \lim_{n \rightarrow \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi \alpha}{\pi}$$

while for $0 < \alpha \leq \frac{1}{2}$ the lim must be replaced by lim inf. However (2.3) does hold when $0 < \alpha \leq \frac{1}{2}$ provided the limit is taken excluding a set of integers having density 0.

These authors did not consider the case $\alpha = 1$ (nor, for that matter, $\alpha = 0$). The appropriate and true conclusion for $\alpha = 1$ is

$$(2.4) \quad \lim_{n \rightarrow \infty} m(n) u_n = 1$$

where, as before,

$$m(n) = \int_0^n (1 - F(x)) dx = \sum_{k=1}^n \sum_{j=k}^{\infty} f_j \sim \sum_{j=1}^n j f_j, \quad n \rightarrow \infty.$$

The proof of (2.3) and (2.4) starts with the following representation for u_n (see [5] or [8, pp. 98–99]): let $\phi(\theta) = \sum f_k e^{ik\theta}$ and put $W(\theta) = \operatorname{Re} [1 - \phi(\theta)]^{-1}$ then provided F has an infinite mean

$$(2.5) \quad u_n = \frac{1}{\pi} \operatorname{Re} \int_0^\pi \frac{e^{-in\theta}}{1 - \phi(\theta)} d\theta = \frac{2}{\pi} \int_0^\pi W(\theta) \cos n\theta d\theta$$

for $n \geq 1$. (When the mean μ is finite (2.5) holds with u_n replaced by $u_n - 1/\mu$.) The lack of a similar formula for $U(t+h) - U(t)$ when F is nonarithmetic constitutes the chief difficulty in the proof of Theorem 1.

Here is a brief proof of (2.4): from (2.5)

$$\frac{\pi}{2} u_n = \left(\int_0^{B/n} + \int_{B/n}^{\pi/2} \right) W(\theta) \cos n\theta d\theta = J_1 + J_2.$$

As in the latter part of the proof of Theorem 1, see (5.10) and (5.11), we get

$$\lim_{n \rightarrow \infty} m(n)J_1 = \pi/2, \quad \limsup_{n \rightarrow \infty} m(n)|J_1| = O(1/B).$$

(The first limit follows directly from Lemma 4, $\alpha = 1$.) Hence

$$\lim_{n \rightarrow \infty} m(n)u_n = \lim_{B \rightarrow \infty} \lim_{n \rightarrow \infty} (2/\pi)m(n)(J_1 + J_2) = 1.$$

J. A. Williamson [11] has extended the results of Lamperti and Garsia [5] to include distributions not necessarily restricted to the positive integers nor to 1-dimension. He does not, however, consider nonarithmetic distributions. He also gives examples showing that (2.3) and its generalization to d -dimensions cannot hold when $\alpha \leq d/2$ without making further assumptions on F . In this connection, see also [5, §3.4].

(iii) Suppose the positive function z on $(0, \infty)$ is nondecreasing and regularly varying with exponent $\beta > 0$. Consider the integral

$$U^*z(t) = \int_0^t z(t-x)U\{dx\} = \int_0^1 z(t(1-y))U\{tdy\}.$$

By Theorem 5 $U(ty)/U(t) \rightarrow y^\alpha$ and it follows that the measure $U\{tdy\}/U(t)$ converges weakly as $t \rightarrow \infty$ to the measure with density $\alpha y^{\alpha-1}$. Furthermore

$$(2.6) \quad f_t(y) = z(t(1-y))/z(t) \rightarrow (1-y)^\beta, \quad t \rightarrow \infty$$

and the convergence is *uniform* in y , $0 \leq y \leq 1$, since each $f_t(y)$ is monotone in y and the limit function $(1-y)^\beta$ is continuous. We see therefore that

$$(2.7) \quad \frac{U^*z(t)}{z(t)U(t)} = \int_0^1 \frac{z(t(1-y))}{z(t)} \cdot \frac{U\{tdy\}}{U(t)} \rightarrow \alpha \int_0^1 (1-y)^\beta y^{\alpha-1} dy$$

as $t \rightarrow \infty$. Now $tz(t) \sim (1+\beta) \int_0^t z(x) dx$ by Karamata's theorem on regular variation, [3, p. 273]. Hence using (2.1) we see that (2.7) may be put in the equivalent form

$$(2.8) \quad \int_0^t z(t-x)U\{dx\} \sim \frac{D(\alpha, \beta)}{m(t)} \int_0^t z(x) dx, \quad t \rightarrow \infty,$$

where

$$D(\alpha, \beta) = \frac{\alpha(1+\beta)}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \int_0^1 (1-y)^\beta y^{\alpha-1} dy = \frac{\Gamma(2+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(2-\alpha)}.$$

Notice that the proof of (2.7) and (2.8) did not depend on the renewal nature, (1.1), of U ; (2.8) remains true when $U > 0$ is any nondecreasing function regularly varying with exponent α , $0 < \alpha \leq 1$, and m is any function satisfying (2.1).

J. Teugels [10] gave a proof of (2.8) when $z > 0$ is nonincreasing and regularly varying with exponent β where $-1 < \beta \leq 0$. The proof is much complicated by the fact that convergence in (2.6) is no longer uniform: when $\beta < 0$ the function

$(1-y)^\beta$ is not bounded at $y=1$. (Teugels imposes a supplementary and rather technical condition on U , in addition to regular variation, which seems to me to be unnecessary; compare the proof in Feller [3, p. 447], of a result where similar problems arise.) Again the proof makes no use of the renewal properties of U .

The regular variation of z with exponent $\beta > -1$ and to a lesser extent the monotonicity of z is clearly essential to the proof of (2.8). In particular, the condition $\beta > -1$ cannot be dropped. When $\beta > -1$, the integral $\int_0^t z(x) dx$ occurring in (2.8) diverges to ∞ as $t \rightarrow \infty$, while for $\beta < -1$, $\int_A^\infty z(x) dx$ is finite for all large enough A . In this case, $\beta < -1$, Theorem 3, §1, usually applies and leads to results directly opposed to (2.8). For example, let $z(t) = t^{-5}$, $t > 1$ and $z(t) = 1$, $t \leq 1$ (z is regularly varying with exponent $\beta = -5$). Then $\int_0^\infty z(x) dx = 5/4$ and, provided $\alpha > \frac{1}{2}$, Theorem 3 gives $m(t)U^*z(t) \rightarrow C_\alpha 5/4 < \infty$ as $t \rightarrow \infty$. On the other hand, if (2.8) were true we would get $m(t)U^*z(t) \rightarrow D(\alpha, -5)5/4 = \infty$.

One last remark. As noted before, one could prove Theorem 5 from Theorem 1 (and Lemma 1) at least for $\frac{1}{2} < \alpha \leq 1$. Since (2.8) depends only on Theorem 5 for the regular variation of U and since Theorem 3 is equivalent to Theorem 1, we see that (2.8) could be derived from Theorem 3, at least in principle, when the only data given, besides the function z , is that U is the renewal function of a distribution F of the form (1.2). In no way, however, can Theorem 3 be proved from (2.8).

(iv) The classical "strong" and "weak" renewal theorems assert respectively

$$(2.9) \quad U(t+h) - U(t) \rightarrow h/\mu \quad (h > 0)$$

$$(2.10) \quad (1/t)U(t) \rightarrow 1/\mu$$

as $t \rightarrow \infty$, for any (nonarithmetic) distribution F on $(0, \infty)$ with mean $\mu \leq \infty$ ($1/\mu$ is interpreted as 0 when $\mu = \infty$). Since $m(t) \rightarrow \mu$ as $t \rightarrow \infty$ we may rewrite (2.9) and (2.10) as

$$U(t+h) - U(t) \sim h/m(t), \quad U(t) \sim t/m(t)$$

provided $\mu < \infty$. Thus apart from the constant C_α in (1.4) and $[\Gamma(\alpha+1)\Gamma(2-\alpha)]^{-1} = C_\alpha/\alpha$ in (2.1), Theorems 1 and 5 are the natural generalizations of these classical theorems.

(v) It should be pointed out that when $\alpha=1$ in (1.2), i.e., if F has the form $1-F(t)=L(t)/t$ for some slowly varying L , then F may or may not have a finite mean. For an example when $\mu < \infty$ consider $L(t) = [\log(t+2)]^{-3} \sim (\log t)^{-3}$. For $\mu = \infty$, consider $L(t) \sim \text{const} > 0$.

As noted in (iv), the classical theorems already imply Theorem 1 (and 5) when $\mu < \infty$. Hence we shall assume from now on that $\mu = \infty$ when $\alpha=1$ in (1.2).

3. Properties of distributions satisfying (1.2). Let F be of the form (1.2) (when $\alpha=1$ we assume in addition that F have infinite expectation, see §2). Let ϕ be the characteristic function of F :

$$\phi(\theta) = \int_0^\infty e^{i\theta x} F\{dx\}.$$

LEMMA 1. The function m defined by (1.3) is regularly varying with exponent $1-\alpha$, and as $t \rightarrow \infty$

$$(3.1) \quad t(1-F(t))/m(t) = t^{1-\alpha}L(t)/m(t) \rightarrow 1-\alpha.$$

We shall need the following immediate consequence of Lemma 1: let $\eta > 0$, then provided $\alpha > 1/2$ and $B > 0$,

$$(3.2) \quad \lim_{t \rightarrow \infty} t^{-1}m^2(t) \int_{\eta}^{t/B} m^{-2}(x) dx = [(2\alpha-1)B^{2\alpha-1}]^{-1}.$$

NOTE. The restriction to $\alpha > 1/2$ in (3.2) partly explains the failure (at least of the proof) of Theorems 1 and 3 when $\alpha \leq 1/2$. See equation (5.11).

Proof. This lemma is a direct consequence of Karamata's theorem on regularly varying functions, see Feller [3, p. 273]. The relation (3.2) likewise follows from this theorem. To see this, define $Z(x) = m^{-2}(x)$ for $x \geq \eta$, $Z(x) = 0$, $0 \leq x < \eta$. Since m is regularly varying with exponent $1-\alpha$, Z varies regularly with exponent $-2(1-\alpha) = 2\alpha-2$. Hence, according to the theorem,

$$\lim_{t \rightarrow \infty} \frac{tZ(t)}{\int_0^t Z(x) dx} = \lim_{t \rightarrow \infty} \frac{(t/B)Z(t/B)}{\int_0^{t/B} Z(x) dx} = 1 + 2\alpha - 2 = 2\alpha - 1.$$

But $Z(t/B) \sim (1/B)^{2\alpha-2}Z(t)$, $t \rightarrow \infty$ (by definition of regular variation). Therefore

$$\int_{\eta}^{t/B} m^{-2}(x) dx \sim (2\alpha-1)^{-1}(t/B)Z(t/B) \sim tm^{-2}(t)/(2\alpha-1)B^{2\alpha-1}$$

as $t \rightarrow \infty$ which proves (3.2).

LEMMA 2. As $\theta \rightarrow 0+$

$$(3.3) \quad 1 - \phi(\theta) \sim e^{-i\pi\alpha/2}\Gamma(2-\alpha)\theta m(1/\theta) \quad (\alpha \neq 0).$$

When $\alpha = 1$ we have in addition to (3.3)

$$(3.4) \quad \operatorname{Re}(1 - \phi(\theta)) \sim \frac{1}{2}\pi\theta L(1/\theta), \quad \theta \rightarrow 0+.$$

Proof. Suppose $0 < \alpha < 1$. Then by (3.1) $m(1/\theta) \sim (1-\alpha)^{-1}\theta^{\alpha-1}L(1/\theta)$, $\theta \rightarrow 0+$. Since $\Gamma(2-\alpha)/(1-\alpha) = \Gamma(1-\alpha)$ we see that (3.3) is equivalent to

$$(3.5) \quad 1 - \phi(\theta) \sim e^{-i\pi\alpha/2}\Gamma(1-\alpha)\theta^{\alpha}L(1/\theta), \quad \theta \rightarrow 0+.$$

Stated in this form (3.3) is well known so we omit the proof. See Garsia and Lamperti [5], or Feller [3, Problems 12 and 13, p. 562]. (There is a slight misprint in the latter reference.)

When $\alpha = 1$, (3.3) and (3.4) do not seem to be as well known. Here then is a brief proof. For any A , $\theta > 0$, write

$$1 - \phi(\theta) = \left(\int_0^{A/\theta} + \int_{A/\theta}^{\infty} \right) (1 - e^{i\theta y}) F\{dy\} = J_1 + J_2$$

then

$$|J_2| = \left| \int_{A/\theta}^{\infty} (1 - e^{iy\theta}) F\{dy\} \right| \leq 2(1 - F(A/\theta)),$$

$$J_1 = \int_0^{A/\theta} (1 - e^{iy\theta}) F\{dy\} = -(1 - e^{iA})(1 - F(A/\theta)) - i \int_0^A e^{ix}(1 - F(x/\theta)) dx.$$

But $1 - F(t) = L(t)/t$ with L slowly varying. Hence

$$(3.6) \quad 1 - \phi(\theta) = O\left(\frac{\theta L(A/\theta)}{A}\right) - i \int_0^A e^{ix}(1 - F(x/\theta)) dx.$$

(The bound in the 0 term is ≤ 4 in magnitude.)

We prove (3.3) first. From (3.1) and slow variation of L we get

$$L(A/\theta) \sim L(1/\theta) = o(m(1/\theta)), \quad \theta \rightarrow 0+.$$

Hence from (3.6)

$$(3.7) \quad \lim_{\theta \rightarrow 0+} \frac{1 - \phi(\theta)}{\theta m(1/\theta)} = -i \lim_{\theta \rightarrow 0+} \int_0^A e^{ix} \left(\frac{1 - F(x/\theta)}{\theta m(1/\theta)} \right) dx$$

provided the latter limit exists. Now by Lemma 1 m is slowly varying ($\equiv$ regularly varying with exponent 0); also $m(0)=0$. Hence, the measure Q_θ on $[0, A]$ with distribution function $Q_\theta(y) = m(y/\theta)/m(1/\theta)$ converges weakly as $\theta \rightarrow 0+$ to the measure which assigns unit mass to the origin. Whence, for any continuous g on $[0, A]$

$$\int_0^A g(x) Q_\theta\{dx\} = \int_0^A g(x) \left(\frac{1 - F(x/\theta)}{\theta m(1/\theta)} \right) dx \rightarrow g(0)$$

as $\theta \rightarrow 0+$. Taking $g(x) = e^{ix}$ we see that the right-hand side of (3.7) equals $-i$. This proves (3.3).

NOTE. The preceding proof requires only minor changes to apply in the case $0 < \alpha < 1$. In particular, a term $O(1/A^\alpha)$ must be added to the right side of (3.7); also Q_θ converges to the measure with density $(1-\alpha)x^{-\alpha}$. In (3.7) one lets $\theta \rightarrow 0+$ followed by $A \rightarrow \infty$. The remainder of the proof is then an evaluation of an improper integral.

To prove (3.4), take real parts in (3.6). Then

$$\frac{\operatorname{Re}(1 - \phi(\theta))}{\theta L(1/\theta)} = O\left(\frac{1}{A}\right) + \int_0^A \frac{\sin x}{x} \cdot \frac{L(x/\theta)}{L(1/\theta)} dx.$$

(The bound in the 0 term is ≤ 8 for all $0 < \theta \leq \theta_A$ sufficiently small.) Letting $\theta \rightarrow 0+$ and then $A \rightarrow \infty$ we see that

$$(3.8) \quad \lim_{\theta \rightarrow 0} \frac{\operatorname{Re}(1 - \phi(\theta))}{\theta L(1/\theta)} = \lim_{A \rightarrow \infty} \lim_{\theta \rightarrow 0} \int_0^A \frac{\sin x}{x} \cdot \frac{L(x/\theta)}{L(1/\theta)} dx$$

provided the iterated limit exists. Since L is slowly varying, we get from the Karamata theorem mentioned earlier

$$\int_0^t L(u) du \sim tL(t), \quad t \rightarrow \infty.$$

Hence, for every $y \geq 0$,

$$\lim_{\theta \rightarrow 0} \int_0^y \frac{L(x/\theta)}{L(1/\theta)} dx = \lim_{\theta \rightarrow 0} \frac{\theta}{L(1/\theta)} \int_0^{y/\theta} L(u) du = y.$$

That is, the measure with density $L(x/\theta)/L(1/\theta)$, $x \geq 0$, converges weakly as $\theta \rightarrow 0$ to Lebesgue measure. Hence for any continuous function f and any compact interval $[0, A]$, say,

$$\lim_{\theta \rightarrow 0} \int_0^A f(x) \left(\frac{L(x/\theta)}{L(1/\theta)} \right) dx = \int_0^A f(x) dx.$$

Letting $f(x) = (\sin x)/x$ and returning to (3.8) we have

$$\lim_{\theta \rightarrow 0+} \frac{\operatorname{Re}(1 - \phi(\theta))}{\theta L(1/\theta)} = \lim_{A \rightarrow \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2},$$

which proves (3.4).

For the purposes of the next two lemmas put

$$(3.9) \quad W(x) = \operatorname{Re} \left(\frac{1}{1 - \phi(x)} \right) = \frac{\operatorname{Re}(1 - \phi(x))}{|1 - \phi(x)|^2}.$$

Note that W is positive since $\operatorname{Re}(1 - \phi(x)) = \int_0^\infty (1 - \cos xt) F\{dt\} > 0$, and symmetric: $W(-x) = W(x)$. Also, W is unbounded (hence undefined) at all x for which $\phi(x) = 1$ (in particular at $x = 0$); at all other x W is continuous.

LEMMA 3. As $\theta \rightarrow 0+$

$$(3.10) \quad \int_0^\theta W(x) dx \sim \frac{\cos(\pi\alpha/2)}{(1-\alpha)\Gamma(2-\alpha)} \cdot \frac{1}{m(1/\theta)}.$$

When $\alpha = 1$ the constant on the right is replaced by

$$\frac{\pi}{2} \quad \left(= \lim_{\alpha \rightarrow 1} \frac{\cos(\pi\alpha/2)}{(1-\alpha)\Gamma(2-\alpha)} \right).$$

REMARK. The integrability of W over bounded intervals containing the origin is, of course, part of the conclusion. This fact, however, is true for any distribution on $(0, \infty)$ (and for some distributions on the entire line); see [3, p. 578].

Proof. A simple calculation using (3.9) and the asymptotic relations (3.3), (3.4) and (3.5) gives

$$(3.11) \quad W(x) \sim \frac{k_\alpha L(1/x)}{x^{2-\alpha} m^2(1/x)}, \quad x \rightarrow 0+,$$

where k_α is the constant occurring on the right in (3.10) ($k_1 = \pi/2$). Next note that the function $1/m(1/x)$, $x > 0$ is absolutely continuous on any interval bounded away from 0 and ∞ . So, by the chain rule and (1.2)

$$(3.12) \quad \frac{d}{dx} \left(\frac{1}{m(1/x)} \right) = \frac{1 - F(1/x)}{x^2 m^2(1/x)} = \frac{L(1/x)}{x^{2-\alpha} m^2(1/x)}$$

for almost all x . (The exceptional set is at most countable.)

Consider $0 < \varepsilon < 1$ fixed but arbitrary. By (3.11) there is a $\lambda = \lambda(\varepsilon) > 0$ such that

$$W(x) \geq (1 \pm \varepsilon) k_\alpha \cdot \frac{L(1/x)}{x^{2-\alpha} m^2(1/x)}$$

whenever $0 < x \leq \lambda$. Integrating these inequalities from $x = \delta$ to $x = \theta$ and using (3.12) yields

$$\int_\delta^\theta W(x) dx \leq (1 \pm \varepsilon) k_\alpha \left(\frac{1}{m(1/\theta)} - \frac{1}{m(1/\delta)} \right)$$

for $0 < \delta \leq \theta \leq \lambda$. Now let $\delta \rightarrow 0$, then $m(1/\delta) \rightarrow \infty$ ($\mu = \infty$ recall), hence

$$(1 - \varepsilon) \frac{k_\alpha}{m(1/\theta)} < \int_0^\theta W(x) dx < (1 + \varepsilon) \frac{k_\alpha}{m(1/\theta)}$$

whenever $0 < \theta \leq \lambda$. This concludes the proof.

By Lemmas 1 and 3, as $t \rightarrow \infty$

$$(3.13) \quad \frac{m(t)}{t} \int_0^\theta W(y/t) dy = m(t) \int_0^{\theta/t} W(x) dx \rightarrow k_\alpha \theta^{1-\alpha}$$

for all $\theta > 0$ and it follows that the measure with density $q_t(y) = (m(t)/t)W(y/t)$ converges weakly as $t \rightarrow \infty$ to a measure which when $\alpha = 1$ is concentrated at the origin with total mass $k_1 = \pi/2$ and when $0 < \alpha < 1$ is absolutely continuous with density $(1 - \alpha)k_\alpha y^{-\alpha}$. Denote the limit measure by E_α . Then for any function f continuous on a compact interval, $[0, B]$, say,

$$m(t) \int_0^{B/t} f(t\theta) W(\theta) d\theta = \int_0^B f(y) q_t(y) dy \rightarrow \int_0^B f(y) E_\alpha\{dy\}, \quad t \rightarrow \infty.$$

Taking $f(y) = \cos y$ we have

LEMMA 4. Let W be given by (3.9). Then for any $B > 0$

$$(3.14) \quad \lim_{t \rightarrow \infty} m(t) \int_0^{B/t} W(\theta) \cos t\theta d\theta = \frac{\cos(\pi\alpha/2)}{\Gamma(2-\alpha)} \int_0^B \frac{\cos y}{y^\alpha} dy, \quad \alpha \neq 1, \\ = \pi/2, \quad \alpha = 1.$$

LEMMA 5. (i) For all $\theta_1 \neq \theta_2$

$$(3.15) \quad |\phi(\theta_2) - \phi(\theta_1)| \leq 2|\theta_2 - \theta_1| m(1/|\theta_2 - \theta_1|).$$

(ii) If F is nonarithmetic, then for each $A > 0$, there is a number $k > 0$, which may depend on A , such that

$$(3.16) \quad \theta m(1/\theta) \leq k|1 - \phi(\theta)| \quad \text{for } 0 < \theta \leq A.$$

If F is arithmetic with span h , (3.16) is true provided $A < 2\pi/h = \text{period of } \phi$.

Proof. (i) Fix $B > 0$. Then

$$\begin{aligned} |\phi(\theta_2) - \phi(\theta_1)| &= \left| \left(\int_0^B + \int_B^\infty \right) (e^{ix\theta_2} - e^{ix\theta_1}) F\{dx\} \right| \\ &\leq \int_0^B |e^{ix\theta_2} - e^{ix\theta_1}| F\{dx\} + 2(1 - F(B)) \\ &\leq |\theta_2 - \theta_1| \int_0^B x F\{dx\} + 2(1 - F(B)). \end{aligned}$$

But $0 \leq \int_0^B x F\{dx\} = m(B) - B(1 - F(B))$ by (1.3). Hence setting $B = |\theta_2 - \theta_1|^{-1}$ we get $|\phi(\theta_2) - \phi(\theta_1)| \leq B^{-1}[m(B) - B(1 - F(B))] + 2(1 - F(B)) = B^{-1}m(B) + 1 - F(B) \leq 2B^{-1}m(B)$ which proves (3.15). (Note that (1.2) was not used; (3.15) holds for any F on $[0, \infty)$.)

(ii) If F is nonarithmetic then $|1 - \phi(\theta)| > 0$ for all $\theta \neq 0$. By Lemma 2 as $\theta \rightarrow 0 +$

$$\theta m(1/\theta)/|1 - \phi(\theta)| \rightarrow 1/\Gamma(2 - \alpha)$$

and it follows that the function

$$\begin{aligned} \beta(\theta) &= \theta m(1/\theta)|1 - \phi(\theta)|^{-1}, \quad \theta \neq 0 \\ &= (\Gamma(2 - \alpha))^{-1}, \quad \theta = 0 \end{aligned}$$

is continuous on $[0, A]$. Taking $k = \max \{\beta(\theta) : 0 \leq \theta \leq A\}$ gives (3.16).

4. An inversion formula for the renewal measure. Define the symmetric renewal measure

$$V\{I\} = \frac{1}{2}(U\{I\} + U\{-I\})$$

where U is given by (1.1) and $-I = \{x : -x \in I\}$. In this section we establish the following

FORMULA. Suppose F is nonarithmetic and has an infinite mean. Then for any continuous function g with compact support whose Fourier transform

$$(4.1) \quad \gamma(x) = \int_{-\infty}^{\infty} e^{ix\theta} g(\theta) d\theta$$

satisfies

$$(4.2) \quad \gamma(x) = O(1/x^2), \quad |x| \rightarrow \infty,$$

we have

$$(4.3) \quad \int_{-\infty}^{\infty} e^{-ix\lambda} \gamma(x) V\{t+dx\} = \int_{-\infty}^{\infty} e^{-it\theta} g(\theta + \lambda) \operatorname{Re} \left(\frac{1}{1 - \phi(\theta)} \right) d\theta$$

for all real λ and t . Here, as elsewhere, ϕ is the characteristic function of F . Note that the integral on the right in (4.3) only extends over a bounded interval. For examples of g and γ see §5.

LEMMA 6. *Let γ be any continuous function satisfying (4.2). Then for every t the integral*

$$\int_{-\infty}^{\infty} |\gamma(x-t)| V\{dx\}$$

is finite.

Proof. Since $\int_{-1}^1 |\gamma(x-t)| V\{dx\} < \infty$ and since $|\gamma(x-t)|$ is bounded by a constant (which may depend on t but not x) times $1/x^2$, it suffices to show

$$(4.4) \quad \int_{|x| \geq 1} \frac{1}{x^2} V\{dx\} = \int_1^{\infty} \frac{1}{x^2} U\{dx\} < \infty.$$

From (2.10) it follows that $U(x) \leq k_1 x$ for some constant $k_1 < \infty$ and all $x \geq 1$. Therefore integrating by parts in (4.4) we get

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} U\{dx\} &= \lim_{A \rightarrow \infty} \left(\frac{U(A)}{A^2} - U(1) + 2 \int_1^A \frac{U(x)}{x^3} dx \right) \\ &= -U(1) + 2 \int_1^{\infty} \frac{U(x)}{x^3} dx \leq 2k_1 \int_1^{\infty} \frac{1}{x^2} dx < \infty \end{aligned}$$

which proves (4.4) and the lemma.

For $0 \leq s < 1$ let V_s be the finite symmetric measure

$$V_s\{dx\} = \frac{1}{2} \sum_{n=0}^{\infty} s^n (F^n\{dx\} + F^n\{-dx\})$$

and note that

$$(4.5) \quad V_s\{I\} \uparrow V\{I\} \quad \text{as } s \uparrow 1$$

for every measurable I bounded or not.

Since

$$\phi(-\theta) = \overline{\phi(\theta)}$$

we have

$$\int_{-\infty}^{\infty} e^{ix\theta} V_s\{dx\} = \frac{1}{2} \sum_{n=0}^{\infty} s^n (\phi^n(\theta) + \phi^n(-\theta)) = \operatorname{Re} \left(\frac{1}{1 - s\phi(\theta)} \right)$$

and an application of Fubini's theorem gives

$$\int_{-\infty}^{\infty} \gamma(x) V_s\{dx\} = \int_{-\infty}^{\infty} g(\theta) \operatorname{Re} \left(\frac{1}{1 - s\phi(\theta)} \right) d\theta \quad (0 \leq s < 1)$$

for any (Lebesgue) integrable function g with γ given by (4.1). Replacing g by

$$g_1(\theta) = e^{-i\theta\lambda} g(\theta + \lambda)$$

and γ by

$$\gamma_1(x) = \int_{-\infty}^{\infty} e^{ix\theta} g_1(\theta) d\theta = e^{-i\lambda(x-t)} \gamma(x-t)$$

we get

$$(4.6) \quad \int_{-\infty}^{\infty} e^{-i\lambda(x-t)} \gamma(x-t) V_s\{dx\} = \int_{-\infty}^{\infty} e^{-it\theta} g(\theta + \lambda) \operatorname{Re} \left(\frac{1}{1-s\phi(\theta)} \right) d\theta.$$

LEMMA 7. *For any continuous function h with compact support*

$$(4.7) \quad \lim_{s \rightarrow 1-} \int_{-\infty}^{\infty} h(\theta) \operatorname{Re} \left(\frac{1}{1-s\phi(\theta)} \right) d\theta = \int_{-\infty}^{\infty} h(\theta) \operatorname{Re} \left(\frac{1}{1-\phi(\theta)} \right) d\theta$$

provided F is nonarithmetic and has infinite expectation.

Proof. We base the proof on the following proposition due to Feller and Orey [4]:

PROPOSITION. *The measure whose density is*

$$\frac{1}{1+\theta^2} \operatorname{Re} \left(\frac{1}{1-s\phi(\theta)} \right)$$

converges weakly and in variation to a finite measure as $s \rightarrow 1-$. In every interval excluding the origin the limit measure is automatically absolutely continuous with density given by

$$\frac{1}{1+\theta^2} \operatorname{Re} \left(\frac{1}{1-\phi(\theta)} \right).$$

If β is the mass assigned to the origin by the limit then $\beta = \pi/\mu > 0$ when μ (the mean of F) is finite and $\beta = 0$ in case $\mu = \infty$.

We omit the proof. (Besides the Feller-Orey paper, see also Breimann [1, p. 221], and Feller [3, p. 578].) The proposition implies, among other things, that

$$\lim_{s \rightarrow 1-} \int_{-\infty}^{\infty} \frac{f(\theta)}{1+\theta^2} \operatorname{Re} \left(\frac{1}{1-s\phi(\theta)} \right) d\theta = \beta f(0) + \int_{-\infty}^{\infty} \frac{f(\theta)}{1+\theta^2} \operatorname{Re} \left(\frac{1}{1-\phi(\theta)} \right) d\theta$$

for every continuous function f with compact support. In our case $\beta = 0$, and (4.7) follows by setting $f(\theta) = (1+\theta^2)h(\theta)$.

Proof of formula (4.3). The very strong convergence (4.5) of the measures V_s to V implies

$$(4.8) \quad \lim_{s \rightarrow 1-} \int_{-\infty}^{\infty} f(x) V_s\{dx\} = \int_{-\infty}^{\infty} f(x) V\{dx\}$$

for every f integrable with respect to V . (In fact, if f is nonnegative the integral on the left is nondecreasing as a function of s and one can show (4.8) holds even if f is not integrable.)

Suppose now g and γ satisfy (4.1) and (4.2) with g continuous and vanishing off a compact set. Then by Lemma 6

$$e^{-i\lambda(x-t)}\gamma(x-t)$$

is integrable with respect to $V\{dx\}$ for every t and λ . Hence by (4.6) and (4.8)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma(x)V\{t+dx\} &\equiv \int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\gamma(x-t)V\{dx\} \\ &= \lim_{s \rightarrow 1-} \int_{-\infty}^{\infty} e^{-is\theta}g(\theta+\lambda) \operatorname{Re} \left(\frac{1}{1-s\phi(\theta)} \right) d\theta.\end{aligned}$$

Formula (4.3) now follows from Lemma 7.

5. Proof of Theorem 1.

1°. Introduce measures μ_t , $t > 0$, by

$$(5.1) \quad \mu_t\{I\} = 2m(t)V\{I+t\} = m(t)(U\{I+t\} + U\{-I-t\})$$

where I is measurable and $I+t = \{x : x-t \in I\}$. Since U is concentrated on $[0, \infty)$ it follows by taking $I = [0, h]$ in (5.1) that

$$U(t+h) - U(t) = (1/m(t))\mu_t\{I\}.$$

Therefore to prove Theorem 1 it suffices to show

$$(5.2) \quad \mu_t\{I\} \rightarrow C_a|I|, \quad t \rightarrow \infty,$$

for every bounded interval I where $|I|$ denotes the length of I and

$$C_a = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}.$$

For each $a > 0$ put $\gamma_a(0) = 1$ and

$$(5.3) \quad \gamma_a(x) = 2(1 - \cos(ax))/a^2x^2.$$

LEMMA 8. Let $\{\mu_t\}$, $t > 0$, be a family of measures such that $\mu_t\{I\} < \infty$ for every compact set I and all t . Suppose for some constant C

$$(5.4) \quad \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\mu_t\{dx\} = C \int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x) dx$$

for every $a > 0$ and all real λ . Then $C^{-1}\mu_t$ converges weakly to Lebesgue measure: $\mu_t\{I\} \rightarrow C|I|$ for every bounded interval I .

(We defer the proof until §6.)

Now γ_a is the Fourier transform (4.1) of the function

$$(5.5) \quad \begin{aligned}g_a(\theta) &= (1/a)(1 - |\theta|/a), & \text{when } |\theta| \leq a \\ &= 0, & \text{when } |\theta| > a.\end{aligned}$$

Whence by the Fourier inversion theorem

$$(5.6) \quad \int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x) dx = 2\pi g_a(\lambda).$$

Clearly we may also apply our inversion formula (4.3) to obtain

$$(5.7) \quad \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) \mu_t\{dx\} = 2m(t) \int_{-\infty}^{\infty} e^{-it\theta} g_a(\theta + \lambda) W(\theta) d\theta$$

where $W(\theta) = \operatorname{Re} [1 - \phi(\theta)]^{-1}$. Note that the integral on the right extends from $\theta = -a - \lambda$ to $\theta = a - \lambda$. From (5.6) and (5.7) we see that (5.4) in our case is equivalent to

$$(5.8) \quad \lim_{t \rightarrow \infty} m(t) \int_{-\infty}^{\infty} e^{-it\theta} g_a(\theta + \lambda) W(\theta) d\theta = \pi C g_a(\lambda)$$

and, by Lemma 8, the proof of (5.2) (and Theorem 1) will be completed when we establish (5.8), with $C = C_a$ for every $a > 0$ and all real λ .

2°. Let $B > 1$ be fixed but otherwise arbitrary, and write the integral in (5.8) as the sum $J_1 + J_2$ where

$$(5.9) \quad \begin{aligned} J_1(t, b) &= \int_{-B/t}^{B/t} e^{-it\theta} g_a(\theta + \lambda) W(\theta) d\theta \quad \text{and} \\ J_2(t, B) &= \int_{|\theta| > B/t} e^{-it\theta} g_a(\theta + \lambda) W(\theta) d\theta \\ &= \int_{B/t}^A [e^{-it\theta} g_a(\theta + \lambda) + e^{it\theta} g_a(\theta - \lambda)] W(\theta) d\theta, \\ A &= \max \{a + \lambda, a - \lambda\}. \end{aligned}$$

(The last integral follows by making the substitution $\theta \rightarrow -\theta$ in the integral $\int_{-\infty}^{-B/t}$, using the evenness of the functions g_a and W and noting that g_a vanishes outside the interval $(-a, a)$.) We will show

$$(5.10) \quad \begin{aligned} \lim_{t \rightarrow \infty} m(t) J_1(t, B) &= g_a(\lambda) \frac{2 \cos \pi\alpha/2}{\Gamma(2-\alpha)} \int_0^B \frac{\cos x}{x^\alpha} dx, & \alpha \neq 1 \\ &= \pi g_a(\lambda), & \alpha = 1 \end{aligned}$$

and

$$(5.11) \quad \limsup_{t \rightarrow \infty} m(t) |J_2(t, B)| = O\left(\frac{1}{B^{2\alpha-1}}\right), \quad \frac{1}{2} < \alpha \leq 1$$

which lead directly to (5.8).

3°. **Proof of (5.10).** It is clear from (5.5) that

$$(5.12) \quad |g_a(\theta_2) - g_a(\theta_1)| \leq (1/a^2) |\theta_2 - \theta_1|$$

for all θ_1, θ_2 . Hence

$$\begin{aligned} m(t) \left| J_1(t, B) - g_a(\lambda) \int_{-B/t}^{B/t} e^{-it\theta} W(\theta) d\theta \right| &\leq m(t) \int_{-B/t}^{B/t} |g_a(\theta + \lambda) - g_a(\lambda)| W(\theta) d\theta \\ &\leq \frac{2B}{a^2} \cdot \frac{m(t)}{t} \int_0^{B/t} W(\theta) d\theta = O\left(\frac{1}{t}\right) \end{aligned}$$

where the $O(1/t)$ follows from (3.10) and Lemma 1. Thus

$$\begin{aligned}\lim_{t \rightarrow \infty} m(t) J_1(t, B) &= g_a(\lambda) \lim_{t \rightarrow \infty} m(t) \int_{-B/t}^{B/t} e^{-it\theta} W(\theta) d\theta \\ &= 2g_a(\lambda) \lim_{t \rightarrow \infty} m(t) \int_0^{B/t} W(\theta) \cos t\theta d\theta\end{aligned}$$

and (5.10) now follows from Lemma 4.

4°. Proof of (5.11). Let

$$\begin{aligned}h_1(\theta) &= e^{-it\theta} g_a(\theta + \lambda) + e^{it\theta} g_a(\theta - \lambda), \\ h_2(\theta) &= e^{-it\theta} g_a(\theta + \pi/t + \lambda) + e^{it\theta} g_a(\theta + \pi/t - \lambda).\end{aligned}$$

Then $h_1(\theta + \pi/t) = -h_2(\theta)$ and making the change of variables $\theta \rightarrow \theta + \pi/t$ in (5.9) gives

$$J_2(t, B) = \int_{B/t}^A h_1(\theta) W(\theta) d\theta = \int_{(B-\pi)/t}^A -h_2(\theta) W(\theta + \pi/t) d\theta$$

(note that the integrand in the last written integral vanishes for $A - \pi/t \leq \theta$). Adding these integrals we get

$$(5.13) \quad 2J_2 = - \int_{(B-\pi)/t}^{B/t} h_2(\theta) W\left(\theta + \frac{\pi}{t}\right) d\theta + \int_{B/t}^A \left[h_1(\theta) W(\theta) - h_2(\theta) W\left(\theta + \frac{\pi}{t}\right) \right] d\theta.$$

Now $|h_j(\theta)| \leq 2/a$ and from (5.12) we have

$$|h_1(\theta) - h_2(\theta)| \leq \left| g_a(\theta + \lambda) - g_a\left(\theta + \lambda + \frac{\pi}{t}\right) \right| + \left| g_a(\theta - \lambda) - g_a\left(\theta - \lambda + \frac{\pi}{t}\right) \right| \leq \frac{2\pi}{a^2 t}.$$

Thus

$$\begin{aligned}\left| h_1(\theta) W(\theta) - h_2(\theta) W\left(\theta + \frac{\pi}{t}\right) \right| &\leq |h_1(\theta) - h_2(\theta)| W(\theta) + \left| W(\theta) - W\left(\theta + \frac{\pi}{t}\right) \right| |h_2(\theta)| \\ &\leq \frac{2\pi}{a^2 t} W(\theta) + \frac{2}{a} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right|.\end{aligned}$$

Applying these inequalities in (5.13) gives

$$\begin{aligned}(5.14) \quad |J_2| &\leq \frac{1}{a} \int_{(B-\pi)/t}^{B/t} W\left(\theta + \frac{\pi}{t}\right) d\theta + \frac{\pi}{a^2 t} \int_{B/t}^A W(\theta) d\theta \\ &\quad + \frac{1}{a} \int_{B/t}^A \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta.\end{aligned}$$

From Lemma 3 it is clear that

$$\lim_{t \rightarrow \infty} m(t) \int_{(B-\pi)/t}^{B/t} W\left(\theta + \frac{\pi}{t}\right) d\theta = k_a[(B + \pi)^{1-\alpha} - B^{1-\alpha}] = O\left(\frac{1}{B^\alpha}\right).$$

Also, since W is integrable on $[0, A]$, $A < \infty$,

$$\frac{\pi}{a^2} \cdot \frac{m(t)}{t} \cdot \int_{B/t}^A W(\theta) d\theta = O\left(\frac{m(t)}{t}\right) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

(That $m(t)/t \rightarrow 0$, $t \rightarrow \infty$, follows from Lemma 1, §3, in our case, but is true for any F on $[0, \infty)$ with m given by (1.3).) Hence from (5.14)

$$\limsup_{t \rightarrow \infty} m(t) |J_2(t, B)| = a^{-1} \limsup_{t \rightarrow \infty} m(t) \int_{B/t}^A \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta + O\left(\frac{1}{B^\alpha}\right).$$

But $O(B^{-\alpha}) = O(B^{1-2\alpha})$ ($B > 1$, $0 \leq \alpha \leq 1$), so the proof of (5.11) will be complete when we show

$$(5.15) \quad \limsup_{t \rightarrow \infty} m(t) \int_{B/t}^A \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta = O\left(\frac{1}{B^{2\alpha-1}}\right).$$

By Lemma 5 (i) we get

$$\begin{aligned} \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| &= \left| \operatorname{Re} \frac{\phi(\theta + \pi/t) - \phi(\theta)}{[1 - \phi(\theta + \pi/t)][1 - \phi(\theta)]} \right| \\ &\leq \frac{2(\pi/t)m(t/\pi)}{|1 - \phi(\theta + \pi/t)||1 - \phi(\theta)|}. \end{aligned}$$

Applying this estimate and the Cauchy-Schwarz inequality to the integral in (5.15) gives

$$\begin{aligned} (5.16) \quad &\int_{B/t}^A \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta \\ &\leq \frac{2\pi}{t} m\left(\frac{t}{\pi}\right) \left(\int_{B/t}^A \frac{d\theta}{|1 - \phi(\theta + \pi/t)|^2} \right)^{1/2} \left(\int_{B/t}^A \frac{d\theta}{|1 - \phi(\theta)|^2} \right)^{1/2} \\ &< 8 \frac{m(t)}{t} \int_{B/t}^{2A} \frac{d\theta}{|1 - \phi(\theta)|^2} \quad (\pi/t \leq A). \end{aligned}$$

Again by Lemma 5(ii) there is a constant $k < \infty$ such that

$$1/|1 - \phi(\theta)| \leq k/\theta m(1/\theta)$$

for $0 < \theta \leq 2A$. Consequently

$$(5.17) \quad \int_{B/t}^{2A} \frac{d\theta}{|1 - \phi(\theta)|^2} \leq k^2 \int_{B/t}^{2A} \frac{d\theta}{\theta^2 m^2(1/\theta)} = k^2 \int_{\eta}^{t/B} \frac{dx}{m^2(x)}$$

where $\eta = 1/2A$. Combining (5.16) and (5.17) we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} m(t) \cdot \int_{B/t}^A \left| W\left(\theta + \frac{\pi}{t}\right) - W(\theta) \right| d\theta &\leq 8k^2 \lim_{t \rightarrow \infty} \frac{m^2(t)}{t} \int_{\eta}^{t/B} \frac{dx}{m^2(x)} \\ &= \frac{1}{(2\alpha-1)B^{2\alpha-1}} \quad (\alpha > \tfrac{1}{2}) \end{aligned}$$

where the last equality comes from (3.2). This completes the proof of (5.15) and hence of (5.11).

5°. The proof of (5.8) with $C = C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$ is now almost immediate. Let

$$\begin{aligned} \Delta(t) &= \left| m(t) \int_{-\infty}^{\infty} e^{-it\theta} g_a(\theta + \lambda) W(\theta) d\theta - \pi C_\alpha g_a(\lambda) \right| \\ &= |m(t)(J_1 + J_2) - \pi C_\alpha g_a(\lambda)| \end{aligned}$$

and suppose $\alpha \neq 1$. Then by (5.10) and (5.11)

$$(5.18) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \Delta(t) &\leq \lim_{t \rightarrow \infty} \left| m(t)J_1 - \frac{\pi g_a(\lambda)}{\Gamma(\alpha)\Gamma(2-\alpha)} \right| + \limsup_{t \rightarrow \infty} m(t)|J_2| \\ &= \frac{g_a(\lambda)}{\Gamma(2-\alpha)} \cdot \left| 2 \cos\left(\frac{\pi\alpha}{2}\right) \int_0^B \frac{\cos x}{x^\alpha} dx - \frac{\pi}{\Gamma(\alpha)} \right| + O\left(\frac{1}{B^{2\alpha-1}}\right). \end{aligned}$$

Now as $B \rightarrow \infty$, $\int_0^B x^{-\alpha} \cos x dx \rightarrow \sin(\pi\alpha/2)\Gamma(1-\alpha)$, hence

$$\lim_{B \rightarrow \infty} \left| 2 \cos\left(\frac{\pi\alpha}{2}\right) \int_0^B \frac{\cos x}{x^\alpha} dx - \frac{\pi}{\Gamma(\alpha)} \right| = \left| \sin(\pi\alpha)\Gamma(1-\alpha) - \frac{\pi}{\Gamma(\alpha)} \right| = 0.$$

Therefore taking the limit in (5.18) as $B \rightarrow \infty$ we get

$$\limsup_{t \rightarrow \infty} \Delta(t) = \lim_{B \rightarrow \infty} \limsup_{t \rightarrow \infty} \Delta(t) = 0$$

which proves (5.8) when $\alpha \neq 1$. When $\alpha = 1$ the proof of (5.8), with $C = C_1 = 1$, from (5.10) and (5.11) is even simpler so we omit it. Theorem 1 now follows from Lemma 8.

6. Proof of Lemma 8. There is no loss in generality in supposing $C = 1$. Taking $\lambda = 0$ in (5.4) and (5.6) we see that as $t \rightarrow \infty$

$$\Delta_t(a) = \int_{-\infty}^{\infty} \gamma_a(x) \mu_t\{dx\} \rightarrow \int_{-\infty}^{\infty} \gamma_a(x) dx = \frac{2\pi}{a} > 0.$$

Hence (5.4) implies that the characteristic function of the probability measure

$$P_t\{dx\} = \frac{1}{\Delta_t(a)} \gamma_a(x) \mu_t\{dx\}$$

converges pointwise to the characteristic function of the probability measure

$$P\{dx\} = (a/2\pi)\gamma_a(x) dx.$$

Consequently, by the continuity theorem for characteristic functions P_t converges weakly to P as $t \rightarrow \infty$. Whence

$$(6.1) \quad \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} B(x) \gamma_a(x) \mu_t\{dx\} = \int_{-\infty}^{\infty} B(x) \gamma_a(x) dx$$

for every bounded continuous function B on R^1 and for every $a > 0$.

For any continuous function f with compact support, write

$$\lambda_t(f) = \int_{-\infty}^{\infty} f(x) \mu_t\{dx\}, \quad \lambda(f) = \int_{-\infty}^{\infty} f(x) dx.$$

Let I be a bounded interval and let $\varepsilon > 0$ be arbitrary but fixed. We can find continuous functions f^+ and f^- both with compact support such that

- (i) $0 \leq f^- \leq 1$, $f^-(x) = 0$ for $x \notin I$,
- (ii) $|I| \leq \lambda(f^-) + \varepsilon$,

(iii) $f^+ \geq 0, f^+(x) = 1$ for x in I ,

(iv) $\lambda(f^+) \leq |I| + \varepsilon$.

Now choose $a > 0$ so small that

$$f^+(x) = f^-(x) = 0 \quad \text{when } |x| \geq \pi/4a.$$

Then since

$$\gamma_a(x) = 2 \left(\frac{1 - \cos ax}{a^2 x^2} \right) > 0 \quad \text{for } |x| < \pi/2a$$

it follows that $B^+ = f^+/\gamma_a$ and $B^- = f^-/\gamma_a$ are continuous functions on R^1 with compact support (hence bounded). Therefore by (6.1)

$$(6.2) \quad \lambda_t(f^\pm) = \int_{-\infty}^{\infty} B^\pm(x) \gamma_a(x) \mu_t\{dx\} \rightarrow \int_{-\infty}^{\infty} B^\pm(x) \gamma_a(x) dx = \lambda(f^\pm).$$

From (i) and (iii) it is clear that

$$\lambda_t(f^-) \leq \mu_t\{I\} \leq \lambda_t(f^+)$$

for all $t > 0$. Letting $t \rightarrow \infty$ and using (6.2) we get

$$\lambda(f^-) \leq \liminf \mu_t\{I\} \leq \limsup \mu_t\{I\} \leq \lambda(f^+),$$

and hence by (ii) and (iv)

$$|I| - \varepsilon \leq \liminf \mu_t\{I\} \leq \limsup \mu_t\{I\} \leq |I| + \varepsilon.$$

Since this holds for every $\varepsilon > 0$ it follows that

$$\mu_t\{I\} \rightarrow |I|, \quad t \rightarrow \infty,$$

which completes the proof.

7. Proof of Theorem 2.

1°. Our first task is to show

$$(7.1) \quad \liminf_{t \rightarrow \infty} m(t)(U(t+h) - U(t)) \geq C_\alpha h \quad (h > 0),$$

or, equivalently,

$$(7.2) \quad \liminf_{t \rightarrow \infty} t^{1-\alpha} L(t)(U(t+h) - U(t)) \geq \frac{\sin \pi\alpha}{\pi} h.$$

(See remark following the statement of Theorem 2.)

Condition (1.2) with $0 < \alpha < 1$ is necessary and sufficient for F to be in the domain of attraction of the unique (apart from a scale factor) stable distribution with exponent α concentrated on $[0, \infty)$. Thus if a sequence $\{B_n\}$ is chosen so that $0 < B_n \uparrow \infty$ and

$$n(1 - F(B_n)) \equiv nB_n^{-\alpha} L(B_n) \rightarrow 1$$

as $n \rightarrow \infty$, then

$$(7.3) \quad F^{n*}(B_n x) \rightarrow \int_0^x q_\alpha(y) dy \quad (n \rightarrow \infty, x \geq 0)$$

where $q_\alpha > 0$ and satisfies

$$\int_0^\infty e^{-\lambda y} q_\alpha(y) dy = \exp[-\lambda^\alpha \Gamma(1-\alpha)], \quad \lambda \geq 0.$$

In addition to (7.3) a local limit theorem for nonarithmetic distributions due to C. Stone [9] implies the somewhat stronger result

$$(7.4) \quad F^{k*}(t+h) - F^{k*}(t) = (h/B_k)q_\alpha(t/B_k) + \delta_k/B_k$$

where $\delta_k \rightarrow 0$ as $k \rightarrow \infty$ uniformly in $t > 0$ ((7.3) only allows $F^{k*}(t+h) - F^{k*}(t) \sim hB_k^{-1}q_\alpha(tB_k^{-1})$ for t and h fixed). Using (7.4) we prove (7.2) almost exactly as Garsia and Lamperti [5] prove the analogous inequality in the arithmetic case. Thus from (1.1) and (7.4)

$$\begin{aligned} U(t+h) - U(t) &> \sum_{k=n}^r (F^{k*}(t+h) - F^{k*}(t)) \\ &= h \sum_n^r \frac{1}{B_k} q_\alpha\left(\frac{t}{B_k}\right) + \sum_n \frac{\delta_k}{B_k}. \end{aligned}$$

Let $0 < A < C < \infty$, and choose $n = [At^\alpha/L(t)]$, $r = [Ct^\alpha/L(t)]$. Then, as in [5], we have both

$$t^{1-\alpha}L(t) \sum_n^r \frac{\delta_k}{B_k} = o(1), \quad t \rightarrow \infty$$

and, writing $x_k = kL(t)/t^\alpha$, $n \leq k \leq r$,

$$\begin{aligned} t^{1-\alpha}L(t) \sum_n^r \frac{1}{B_k} q_\alpha\left(\frac{t}{B_k}\right) &\sim \sum_{A \leq x_k \leq C} x_k^{-1/\alpha} q_\alpha(x_k^{-1/\alpha})(x_{k+1} - x_k) \\ &\rightarrow \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha}) dx \end{aligned}$$

as $t \rightarrow \infty$. Hence for any $\varepsilon > 0$

$$t^{1-\alpha}L(t)(U(t+h) - U(t)) \geq \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha}) dx - \varepsilon$$

for all t sufficiently large. In other words

$$\liminf_{t \rightarrow \infty} t^{1-\alpha}L(t)(U(t+h) - U(t)) \geq \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha}) dx,$$

and (7.2) now follows by letting $A \rightarrow 0$, $C \rightarrow \infty$ and noting

$$\int_0^\infty x^{-1/\alpha} q_\alpha(x^{-1/\alpha}) dx = \alpha \int_0^\infty y^{-\alpha} q_\alpha(y) dy = \frac{\sin \pi\alpha}{\pi}.$$

2°. To complete the proof of Theorem 2 we need the following lemma (also needed in the proof of Theorem 3).

LEMMA 9. Let z be any nonnegative integrable (but not necessarily dri) function on $[0, \infty)$. Then

$$(7.5) \quad \liminf_{t \rightarrow \infty} m(t) \int_0^t z(t-y)U\{dy\} \leq C_\alpha \int_0^\infty z(x) dx \quad (0 < \alpha \leq 1).$$

To finish the proof of Theorem 2 we set $z(x) = 1$ for $0 \leq x \leq h$, $z(x) = 0$ elsewhere. Noting that $m(t+h) \sim m(t)$ as $t \rightarrow \infty$ we get from (7.5)

$$(7.6) \quad \liminf_{t \rightarrow \infty} m(t)(U(t+h) - U(t)) = \liminf_{t \rightarrow \infty} m(t+h)U^*z(t+h) \\ \leq C_\alpha \int_0^\infty z(x) dx = C_\alpha h.$$

Together (7.1) and (7.6) give (1.5).

Proof of Lemma 9. Let $v(t) = U^*z(t) = \int_0^t z(t-x)U\{dx\}$. Then

$$\hat{v}(\lambda) = \int_0^\infty e^{-\lambda x} v(x) dx = \left(\int_0^\infty e^{-\lambda x} z(x) dx \right) \hat{U}(\lambda) = \hat{z}(\lambda) \hat{U}(\lambda)$$

where $\hat{U}$ is defined as in §2(i). Since U is regularly varying with exponent α we have

$$\hat{U}(\lambda) \sim \Gamma(\alpha+1)U(1/\lambda) \quad \text{as } \lambda \rightarrow 0+$$

by Theorem 1 in [3, p. 420]. Now $\hat{z}(0) = \int_0^\infty z(x) dx < \infty$ and it follows that

$$\hat{v}(\lambda) \sim \hat{z}(0)\Gamma(\alpha+1)U(1/\lambda), \quad \lambda \rightarrow 0+$$

which, by the converse of the same Theorem 1 in [3], is the same as

$$(7.7) \quad \int_0^t v(x) dx \sim \hat{z}(0)U(t), \quad t \rightarrow \infty.$$

Now by Theorem 5 in §2

$$(7.8) \quad U(t) \sim (\Gamma(\alpha+1)\Gamma(2-\alpha))^{-1}t/m(t) = (C_\alpha/\alpha)t/m(t)$$

as $t \rightarrow \infty$; also, since $1/m$ is regularly varying with exponent $\alpha-1 > -1$ we have for fixed $\eta > 0$

$$(7.9) \quad \frac{1}{\alpha} \frac{t}{m(t)} \sim \int_\eta^t \frac{dx}{m(x)}, \quad t \rightarrow \infty$$

(cf. [3, p. 273]). From (7.7), (7.8), and (7.9) it follows that

$$(7.10) \quad \int_0^t v(x) dx \sim C_\alpha \hat{z}(0) \int_\eta^t \frac{dx}{m(x)}, \quad t \rightarrow \infty.$$

Suppose, contrary to (7.5),

$$\liminf_{t \rightarrow \infty} m(t)v(t) > C_\alpha \hat{z}(0).$$

Then for some $\varepsilon > 0$ and all $x \geq \eta$ sufficiently large

$$v(x) \geq (1+\varepsilon)C_\alpha \hat{z}(0)(1/m(x)).$$

Hence

$$\int_0^t v(x) dx \geq \int_\eta^t v(x) dx \geq (1+\varepsilon)C_\alpha \hat{z}(0) \int_\eta^t \frac{dx}{m(x)}$$

for all $t \geq \eta$. But this contradicts (7.10).

8. Proof of Theorems 3 and 4.

1°. Let $h > 0$. Throughout this section put $z_k(x) = 1$ when $(k-1)h \leq x < kh$, $z_k(x) = 0$ elsewhere, and let

$$v_k(t) = U^* z_k(t) = U(t - (k-1)h) - U(t - kh).$$

Since $m(t - kh) \sim m(t)$ for fixed kh , $t \rightarrow \infty$, we have by Theorems 1 and 2

$$(8.1) \quad \begin{aligned} \liminf_{t \rightarrow \infty} m(t) v_k(t) &= C_\alpha h & (0 < \alpha \leq \tfrac{1}{2}), \\ \lim_{t \rightarrow \infty} m(t) v_k(t) &= C_\alpha h & (\tfrac{1}{2} < \alpha \leq 1); \quad k = 1, 2, \dots \end{aligned}$$

2°. Let $z \geq 0$ be any dri function on $[0, \infty]$. Then

$$(8.2) \quad \liminf_{t \rightarrow \infty} m(t) \int_0^t z(t-y) U\{dy\} \geq C_\alpha \int_0^\infty z(x) dx \quad (0 < \alpha \leq 1).$$

Theorem 4 follows immediately from (8.2) and Lemma 9.

To prove (8.2) let $\varepsilon > 0$ be arbitrary. We suppose $h > 0$ is so small that

$$\int_0^\infty z(x) dx - \frac{\varepsilon}{C_\alpha} < \sum_1^\infty a_k h$$

where $a_k = \inf \{z(x) : (k-1)h \leq x < kh\}$. Then by (8.1) and Fatou's lemma

$$\begin{aligned} C_\alpha \int_0^\infty z(x) dx - \varepsilon &< \sum_1^\infty a_k \liminf_{t \rightarrow \infty} m(t) v_k(t) \\ &\leq \liminf_{t \rightarrow \infty} m(t) \sum_1^\infty a_k U^* z_k(t) \\ &\leq \liminf_{t \rightarrow \infty} m(t) U^* z(t) \end{aligned}$$

which implies (8.2) as $\varepsilon > 0$ is arbitrary.

3°. From now on in addition to being dri we assume z satisfies (1.7). That is for some constant $b < \infty$

$$(8.3) \quad 0 \leq z(x) \leq b/x, \quad x > 0.$$

We also assume $\frac{1}{2} < \alpha \leq 1$ in (1.2). Obviously our goal now is to show

$$(8.4) \quad \limsup_{t \rightarrow \infty} m(t) \int_0^t z(t-y) U\{dy\} \leq C_\alpha \int_0^\infty z(x) dx.$$

4°. Fix $0 < \theta < 1$. Then

$$(8.5) \quad \limsup_{t \rightarrow \infty} m(t) \int_0^{t\theta} z(t-y) U\{dy\} \leq \frac{b C_\alpha \theta^\alpha}{\alpha(1-\theta)}$$

and

$$(8.6) \quad \limsup_{t \rightarrow \infty} m(t) \int_{t\theta}^t z(t-y) U\{dy\} \leq C_\alpha \int_0^\infty z(x) dx.$$

Proof of (8.5). From (8.3)

$$\int_0^{t\theta} z(t-y)U\{dy\} \leq b \int_0^{t\theta} \frac{1}{t-y} U\{dy\} \leq \frac{b}{(1-\theta)t} U(t\theta).$$

But $U(t\theta) \sim \theta^\alpha U(t) \sim \alpha^{-1} C_\alpha \theta^\alpha (t/m(t))$ as $t \rightarrow \infty$ by Theorem 5 and Lemma 1. Hence

$$\limsup_{t \rightarrow \infty} m(t) \int_0^{t\theta} z(t-y)U\{dy\} \leq \frac{b}{1-\theta} \lim_{t \rightarrow \infty} \frac{m(t)}{t} U(t\theta) = \frac{bC_\alpha \theta^\alpha}{\alpha(1-\theta)}.$$

Proof of (8.6). Let $\varepsilon > 0$ be arbitrary and put $b_k = \sup \{z(x) : (k-1)h \leq x < kh\}$. We assume h is so small that

$$(8.7) \quad \sum_1^\infty b_k h < \int_0^\infty z(x) dx + \frac{\varepsilon}{C_\alpha}.$$

Let n be the largest integer satisfying $(n-1)h \leq t(1-\theta)$. Then $z_k(t-y) = 0$ for $k \geq n+1$ and all $t\theta \leq y \leq t$, hence

$$(8.8) \quad \int_{t\theta}^t z(t-y)U\{dy\} \leq \sum_1^n b_k \int_{t\theta}^t z_k(t-y)U\{dy\} \leq \sum_1^n b_k v_k(t).$$

Suppose for the moment that

$$(8.9) \quad \lim_{t \rightarrow \infty} m(t) \sum_1^n b_k v_k(t) = C_\alpha \sum_1^\infty b_k h.$$

Then by (8.8) and (8.7)

$$\limsup_{t \rightarrow \infty} m(t) \int_{t\theta}^t z(t-y)U\{dy\} \leq C_\alpha \sum_1^\infty b_k h < C_\alpha \int_0^\infty z(x) dx + \varepsilon$$

which yields (8.6) on letting $\varepsilon \rightarrow 0$.

Let $\beta_t(k) = b_k m(t) v_k(t)$ for $k = 1, 2, \dots, n$ and $\beta_t(k) = 0$ for $k \geq n+1$ then $m(t) \sum_1^n b_k v_k(t) = \sum_{k=1}^\infty \beta_t(k)$, and since, by (8.1), $\beta_t(k) \rightarrow C_\alpha h b_k$, $k = 1, 2, \dots$, $t \rightarrow \infty$, we see that to establish (8.9) it will suffice to find numbers T and B so that

$$(8.10) \quad \beta_t(k) \leq B b_k \quad \text{for all } k \geq 1 \text{ and all } t \geq T.$$

First choose s_0 so that $s \geq s_0$ implies

$$U(s+h) - U(s) < 2C_\alpha h/m(s).$$

Next from $m(t\theta - h) \sim m(t\theta) \sim \theta^{1-\alpha} m(t)$ as $t \rightarrow \infty$, we find a t_0 so that for all $t \geq t_0$

$$m(t) < 2\theta^{\alpha-1} m(t\theta - h).$$

Suppose now that $t \geq t_0$, $t\theta - h \geq s_0$ and $1 \leq k \leq n$. Noting that $t\theta - h \leq t - kh$, by definition of n , we get

$$m(t) < 2\theta^{\alpha-1} m(t\theta - h) \leq 2\theta^{\alpha-1} m(t - kh)$$

and

$$v_k(t) = U(t - kh + h) - U(t - kh) < 2C_\alpha h/m(t - kh),$$

that is, $m(t)v_k(t) < 4C_\alpha h\theta^{\alpha-1}$. Since $\beta_i(k)=0$ for $k > n$ we see that (8.10) holds with $T = \max \{(s_0 + h)/\theta, t_0\}$ and $B = 4C_\alpha h\theta^{\alpha-1}$. This completes the proof of (8.6).

5°. From (8.5) and (8.6) we have

$$\begin{aligned}\limsup_{t \rightarrow \infty} m(t)U^*z(t) &= \limsup_{t \rightarrow \infty} m(t) \left(\int_0^{t\theta} + \int_{t\theta}^t \right) z(t-y)U\{dy\} \\ &= O\left(\frac{\theta^\alpha}{1-\theta}\right) + C_\alpha \int_0^\infty z(x) dx\end{aligned}$$

whenever $0 < \theta < 1$. Letting $\theta \rightarrow 0$ gives (8.4).

Theorem 3 is evident from (8.2) and (8.4).

9. An application. In this section we study the asymptotic behavior of the spent and residual waiting times associated with a renewal process whose waiting time distribution has the form (1.2) with $\alpha = 1$.

A renewal process with waiting time distribution F is any sequence $\{S_n\}$, $n \geq 0$ of the form $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geq 1$, where the X_n are positive mutually independent random variables with common distribution F . The S_n are usually interpreted as consecutive points on a time axis and are called renewal epochs. The X_n are then called waiting times. In this context $U\{I\} = \sum F^n\{I\} = \sum P\{S_n \in I\}$ is clearly the expected number of renewal epochs falling in I .

Our interest here is in two auxiliary random variables Y_t and Z_t called, respectively, the spent and residual (or excess) waiting time at epoch t defined as follows: let $N_t = \max \{n : S_n \leq t\}$ (=the number of renewal epochs in $(0, t]$). Then

$$Y_t = t - S_{N_t}, \quad Z_t = S_{N_t+1} - t.$$

When the distribution F has a finite mean, Y_t and Z_t have nondegenerate limit distributions:

$$(9.1) \quad \lim_{t \rightarrow \infty} P\{Y_t > y, Z_t > z\} = \frac{1}{\mu} \int_{y+z}^\infty [1 - F(u)] du$$

(see [3, p. 371, problem 3], or [2, Theorem 1]).

In general when $\mu = \infty$ the most one can say is $Y_t \rightarrow \infty$ and $Z_t \rightarrow \infty$ in probability. However, if F has the form (1.2) with $0 < \alpha < 1$, then Lamperti [7] and Dynkin [2] have shown that Y_t/t and Z_t/t have nontrivial limit distributions:

$$\lim_{t \rightarrow \infty} P\left\{\frac{Y_t}{t} > y, \frac{Z_t}{t} > z\right\} = \frac{\sin \pi \alpha}{\pi} \int_y^1 (z+u)^{-\alpha} (1-u)^{\alpha-1} du,$$

for $0 \leq z < \infty$ and $0 \leq y \leq 1$. See also Feller [3, p. 447]. These writers show that (1.2) with $0 < \alpha < 1$ is in fact necessary and sufficient for Y_t/t and Z_t/t to have nontrivial limit distributions. (Dynkin proves that if $Y_t/\beta(t)$ (or $Z_t/\beta(t)$) has a nontrivial limit distribution where $\beta(t)$ is regularly varying and approaches infinity as $t \rightarrow \infty$, then (1.2) holds for some $0 < \alpha < 1$ and $\beta(t)/t \rightarrow \text{const.}$)

When $\alpha = 1$ in (1.2) F may or may not have a finite mean (see §2(v)), but in either case it is quite straightforward to show that $Y_t/t \rightarrow 0$ and $Z_t/t \rightarrow 0$ in probability

(see (9.4) for the precise rate). But as noted above if $\mu = \infty$ we also have Y_t and $Z_t \rightarrow \infty$ (in probability) so one might expect that some nonlinear normalization such as $\lambda(Y_t)/\beta(t)$ where $\lambda(t), \beta(t) \rightarrow \infty$ will yet produce a nontrivial limit distribution.

THEOREM 6. *Let F have the form*

$$1 - F(t) = L(t)/t, \quad t > 0,$$

where L is slowly varying at ∞ and suppose the mean of F is infinite. Then for $0 \leq a \leq 1, b \geq 0$

$$(9.2) \quad \lim_{t \rightarrow \infty} P\left\{\frac{m(Y_t)}{m(t)} \leq a, \frac{m(Z_t)}{m(t)} \leq b\right\} = \min\{a, b\}$$

where m is the function defined by (1.3).

The limit distribution in (9.2) is just the uniform distribution concentrated on the diagonal of the unit square, consequently we have the following.

COROLLARY. $(m(Y_t) - m(Z_t))/m(t) \rightarrow 0$ in probability as $t \rightarrow \infty$, and for $0 < \theta < 1$

$$(9.3) \quad \lim_{t \rightarrow \infty} P\left\{\frac{m(Y_t)}{m(t)} \leq \theta\right\} = \lim_{t \rightarrow \infty} P\left\{\frac{m(Z_t)}{m(t)} \leq \theta\right\} = \theta.$$

REMARKS. 1. Since Z_t and $Y_t \rightarrow \infty$ in probability it is clear that the function m in these results may be replaced by any function m_1 such that $m_1(t) \uparrow \infty$ and $m_1(t)/m(t) \rightarrow k \neq 0$ as $t \rightarrow \infty$.

2. It should be pointed out that for any F on $(0, \infty)$ with a finite mean (9.3) (but *not* (9.2)) is still valid. To see this consider for example Y_t . Let ρ be the continuous inverse of m : $\rho(m(t)) = t$, $m(\rho(x)) = x$, $0 \leq x < \mu$. From (9.1),

$$\lim_{t \rightarrow \infty} P\{Y_t \leq y\} = \mu^{-1} \int_0^y [1 - F(x)] dx = m(y)/\mu;$$

hence

$$\lim_{t \rightarrow \infty} P\{m(Y_t)/m(t) \leq \theta\} = \lim_{t \rightarrow \infty} P\{Y_t \leq \rho(\theta\mu)\} = m(\rho(\theta\mu))/\mu = \theta \quad (0 < \theta < 1).$$

Our last result gives precise information about the distribution of Y_t/t and Z_t/t for large t .

THEOREM 7. *Let F be as in Theorem 6 and let $0 \leq a \leq 1, b \geq 0, a + b \neq 0$. Then as $t \rightarrow \infty$*

$$(9.4) \quad P\left\{\frac{Y_t}{t} > a, \frac{Z_t}{t} > b\right\} \sim \frac{L(t)}{m(t)} \cdot \log\left(\frac{1+b}{a+b}\right).$$

(Note that $L(t)/m(t) \rightarrow 0$ as $t \rightarrow \infty$ by Lemma 1.)

Proof. From (9.7) it follows that

$$\begin{aligned} G_t(a, b) &= P\{Y_t > ta, Z_t > tb\} = \int_0^{t-at} [1 - F(t+tb-x)] U\{dx\} \\ &= \int_0^{1-a} [1 - F(t(1+b-y))] U\{tdy\}. \end{aligned}$$

We now argue as in the proof of (2.8): By Lemma 1 and Theorem 5 (with $\alpha=1$)

$$[1-F(t)]U(t) \sim L(t)/m(t), \quad t \rightarrow \infty,$$

so

$$G_t(a, b) \frac{m(t)}{L(t)} \sim \int_0^{1-a} \frac{1-F(t(1+b-y))}{1-F(t)} \cdot \frac{U\{tdy\}}{U(t)}, \quad t \rightarrow \infty.$$

Now

$$f_t(y) = \frac{1-F(t(1+b-y))}{1-F(t)} \rightarrow \frac{1}{1+b-y} \quad \text{as } t \rightarrow \infty$$

and the convergence is uniform in $0 \leq y \leq 1-a$ (provided $a+b \neq 0$) since each $f_t(y)$ is monotone in y and since the limit $1/(1+b-y)$ is continuous on $0 \leq y \leq 1-a$. Also, since $U(ty)/U(t) \rightarrow y$, the measure $U\{tdy\}/U(t)$ converges weakly to Lebesgue measure as $t \rightarrow \infty$.

From these remarks we see that

$$P\{Y_t > ta, Z_t > tb\} \frac{m(t)}{L(t)} \rightarrow \int_0^{1-a} \frac{1}{1+b-y} dy, \quad t \rightarrow \infty,$$

and (9.4) follows.

Proof of Theorem 6. Since we use Theorem 1 we shall assume F is nonarithmetic. Theorem 6 is still true when F is arithmetic, and, though certain of the details in the present proof must be slightly modified, the essential points are the same. (Of course one uses (2.4) rather than Theorem 1 in the arithmetic case.)

Let ρ be the strictly increasing continuous inverse of the function m : $\rho(m(t)) = m(\rho(t)) = t$. Since F has infinite expectation, $m(t) \rightarrow \infty$ as $t \rightarrow \infty$ so ρ is defined on $[0, \infty)$. Fix $0 < a < 1$, $b > 0$ and let

$$(9.5) \quad a_t = \rho(am(t)), \quad b_t = \rho(bm(t)).$$

We will prove

$$(9.6) \quad \lim_{t \rightarrow \infty} P\{Y_t \leq a_t, Z_t > b_t\} = \max\{a, b\} - b$$

which is evidently the same as (9.2).

Our starting point in proving (9.6) is the following equation

$$(9.7) \quad P\{Y_t \leq a, Z_t > b\} = \int_{t-a}^t [1-F(t+b-y)]U\{dy\}.$$

Here is a probabilistic derivation: By definition $Y_t = t - S_{N_t}$, $Z_t = S_{N_t+1} - t$ where $N_t = n$ if and only if $S_n \leq t < S_{n+1}$. Hence the joint event $\{Y_t \leq a, Z_t > b\}$ occurs if and only if for some (unique) n , $S_n = y$ with $t-a \leq y \leq t$ and then $Z_t = S_{n+1} - t = X_{n+1} + y - t > b$. By independence of S_n and X_{n+1} , the conditional probability of the second event is simply $P\{X_{n+1} > t+b-y\} = 1-F(t+b-y)$. Multiplying this by $F^{n*}\{dy\}$, the distribution of S_n , and summing over all $t-a \leq y \leq t$ we get

$$P\{Y_t \leq a, Z_t > b, N_t = n\} = \int_{t-a}^t [1-F(t+b-y)]F^{n*}\{dy\}.$$

Summing over all $n \geq 0$ gives (9.7) since $\sum F^n = U$.

LEMMA 10. (i) Let a_t be defined by (9.5) with $0 < a < 1$. Then

$$(9.8) \quad a_t/t \rightarrow 0 \quad \text{but} \quad a_t \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty.$$

(ii) Let $\varepsilon, \delta > 0$. Then there is a $T > 0$ such that for all $t \geq T$ and all $\frac{1}{2}t \leq y \leq 2t$ we have

$$(9.9) \quad \frac{1-\varepsilon}{m(t)} \delta < U(y+\delta) - U(y) < \frac{1+\varepsilon}{m(t)} \delta.$$

(We prove Lemma 10 later.)

Let $\varepsilon, \delta > 0$ with $0 < \varepsilon < 1$ be fixed but arbitrary. By Lemma 10, $a_t \rightarrow \infty$ and $(t-a_t)/t \rightarrow 1$ as $t \rightarrow \infty$. Hence by choosing T_1 sufficiently large we may assume that both (9.9) and the inequalities

$$(9.10) \quad \frac{1}{2}t + 10\delta < t - a_t < t < 2t - 10\delta, \quad a_t > 100\delta,$$

hold simultaneously for all $t \geq T_1$. Let $t \geq T_1$ and consider the partition $0 = y_0 < y_1 < y_2 < \dots$ of $[0, \infty)$ where $y_k = k\delta$. Write

$$\Delta U_k = U(y_{k+1}) - U(y_k) = U(y_k + \delta) - U(y_k)$$

and let y_r and y_n be chosen as in the following diagram

$$(9.11) \quad \begin{array}{ccccccc} & & y_r & & y_{r+1} & & y_{n-1} & & y_n \\ \text{---} & \text{---} & | & \text{---} & | & \text{---} & | & \text{---} & | & \text{---} & \rightarrow \\ 0 & & t-a_t-\delta & & t-a_t & & t & & t+\delta \end{array}$$

($y_r \leq t - a_t$, $y_{n-1} \leq t$). Since $y_r > t - a_t - \delta$ and $y_n < t + \delta$ it follows from (9.9) and (9.10) that

$$(9.12) \quad \frac{1-\varepsilon}{m(t)} \delta < \Delta U_k < \frac{1+\varepsilon}{m(t)} \delta, \quad k = r, r+1, \dots, n-1, n.$$

Now let $f(y) = 1 - F(t + b_t - y)$, $0 \leq y \leq t + b_t$. Then f is nonnegative, nondecreasing and bounded by 1. Consequently by (9.7), (9.11) and (9.12)

$$\begin{aligned} P\{Y_t \leq a_t, Z_t > b_t\} &= \int_{t-a_t}^t f(y) U\{dy\} \leq \sum_{k=r}^{n-1} f(y_{k+1}) \Delta U_k < \frac{1+\varepsilon}{m(t)} \sum_{k=r}^{n-1} f(y_{k+1}) \delta \\ &= \frac{1+\varepsilon}{m(t)} \sum_{k=r+1}^n f(y_k) \delta \leq \frac{1+\varepsilon}{m(t)} \int_{y_{r+1}}^{y_{n+1}} f(y) dy \\ &\leq \frac{1+\varepsilon}{m(t)} \int_{t-a_t}^{t+2\delta} f(y) dy \leq \frac{1+\varepsilon}{m(t)} \int_{t-a_t}^t f(y) dy + \frac{4\delta}{m(t)}. \end{aligned}$$

A similar calculation gives

$$P\{Y_t \leq a_t, Z_t > b_t\} > \frac{1-\varepsilon}{m(t)} \int_{t-a_t}^t f(y) dy - \frac{4\delta}{m(t)}.$$

But

$$\begin{aligned}\int_{t-a_t}^t f(y) dy &= \int_{t-a_t}^t [1 - F(t + b_t - y)] dy = m(a_t + b_t) - m(b_t) \\ &= m(a_t + b_t) - bm(t).\end{aligned}$$

Therefore for all $t \geq T_1$

$$(9.13) \quad P\{Y_t \leq a_t, Z_t > b_t\} \leq (1 \pm \varepsilon) \left(\left(\frac{m(a_t + b_t)}{m(t)} \right) - b \right) \pm \frac{4\delta}{m(t)}.$$

Assume for the moment

$$(9.14) \quad \lim_{t \rightarrow \infty} \frac{m(a_t + b_t)}{m(t)} = \max\{a, b\}.$$

Then since $m(t) \rightarrow \infty$ as $t \rightarrow \infty$ we conclude from (9.13) and (9.14):

$$\begin{aligned}(1 - \varepsilon)(\max\{a, b\} - b) &\leq \liminf P\{Y_t \leq a_t, Z_t > b_t\} \\ &\leq \limsup P\{Y_t \leq a_t, Z_t > b_t\} \\ &\leq (1 + \varepsilon)(\max\{a, b\} - b)\end{aligned}$$

and (9.6) follows.

It remains to prove (9.14). Let $c = \max\{a, b\}$ and $c_t = \rho(cm(t))$. Then $cm(t) = m(c_t) \leq m(a_t + b_t) \leq m(2c_t)$, or

$$(9.15) \quad c \leq m(a_t + b_t)/m(t) \leq m(2c_t)/m(t) = (m(2c_t)/m(c_t))c.$$

Now m is slowly varying by Lemma 1 and $c_t \rightarrow \infty$ by Lemma 10, hence

$$m(2c_t)/m(c_t) \rightarrow 1$$

as $t \rightarrow \infty$. Letting $t \rightarrow \infty$ in (9.15) gives (9.14). This completes the proof of Theorem 6.

Proof of Lemma 10. (i) Since both $\rho(t) \rightarrow \infty$ and $m(t) \rightarrow \infty$ it is clear that $a_t = \rho(am(t)) \rightarrow \infty$ as $t \rightarrow \infty$ for any $a > 0$. Let $0 < a < b$ we show

$$(9.16) \quad \rho(am(t))/\rho(bm(t)) = a_t/b_t \rightarrow 0, \quad t \rightarrow \infty.$$

To get (9.8) take $b = 1$, $0 < a < 1$ in (9.16).

Suppose (9.16) fails. Then for some $0 < \theta < 1$ and some sequence $t_n \rightarrow \infty$ we have $\theta \leq a_{t_n}/b_{t_n} \leq 1$ for all n . Hence $m(\theta b_{t_n}) \leq m(a_{t_n}) < m(b_{t_n})$, or since $m(a_t) = am(t)$, $m(b_t) = bm(t)$,

$$(9.17) \quad m(\theta b_{t_n})/m(b_{t_n}) \leq a/b < 1.$$

But $m(\theta b_{t_n})/m(b_{t_n}) \rightarrow 1$ as $t_n \rightarrow \infty$, since m is slowly varying and $b_{t_n} \rightarrow \infty$, so (9.17) leads to the contradiction $1 \leq a/b < 1$. Hence (9.16) must be true.

(ii) Let $\varepsilon, \varepsilon_1, \varepsilon_2, \delta$ be positive numbers with $\varepsilon_1, \varepsilon_2 < 1$. Since m is slowly varying there is a $t_1 > 0$ such that

$$(9.18) \quad 1 - \varepsilon_1 < m(t/2)/m(2t) < 1 + \varepsilon_1 \quad \text{for all } t \geq t_1.$$

By Theorem 1, $\alpha=1$, we can find $t_2>0$ so that

$$(9.19) \quad (1-\varepsilon_2) \cdot \frac{\delta}{m(y)} < U(y+\delta) - U(y) < (1+\varepsilon_2) \cdot \frac{\delta}{m(y)}, \quad \text{for } y \geq t_2.$$

Suppose now that $\frac{1}{2}t \geq \max\{t_1, t_2\}$ and $\frac{1}{2}t \leq y \leq 2t$. Then since m is increasing

$$m(t/2)/m(2t) \leq m(t)/m(y) \leq m(2t)/m(t/2).$$

Consequently $1-\varepsilon_1 < m(t)/m(y) < 1/(1-\varepsilon_1)$ by (9.18), and from (9.19) it follows that

$$(1-\varepsilon_1)(1-\varepsilon_2) \frac{\delta}{m(t)} < U(y+\delta) - U(y) < \left(\frac{1+\varepsilon_2}{1-\varepsilon_1} \right) \frac{\delta}{m(t)}.$$

By (pre) choosing $\varepsilon_1, \varepsilon_2$ so that $(1-\varepsilon_1)(1-\varepsilon_2) \geq 1-\varepsilon$ and $(1+\varepsilon_2)/(1-\varepsilon_1) \leq 1+\varepsilon$ we get (9.9) with $T = \max\{2t_1, 2t_2\}$.

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